

# Quantum Inflation? \*

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## Abstract

We consider curved space quantum corrections to the equations of motion of the inflaton field in the early Universe. Using the stochastic formalism in phase space, we demonstrate that the quantum corrected evolution of the inflaton can differ dramatically from its classical evolution when the mass scales in the potential become large, which is naturally the case in fundamental theories describing Planck scale physics. Using the example of the cosine potential, we show that the prolonged, perhaps even eternal, quantum-inflationary period is expected with a significant probability. This feature of Planck scale potentials can be dangerous, but it offers also a possibility of creating the inflationary phase in string or supergravity models, where no natural realization of inflation has been found so far.

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The cosmological standard model demands an unacceptable amount of fine tuning concerning the initial conditions of the Universe. A superluminal expansion of the scale factor, appearing during an inflationary epoch at a very early stage of the evolution of the Universe, was thought to circumvent these unnatural features [1]. However, the classical analysis of most of the proposed inflationary models shows that also inflation demands a fine tuning of either the initial state (new inflation) or the parameters of the potential (nearly every model). One of the reasons is the need for flat potentials coming from the fact that during inflation at least 60 e-folds must be attained to solve the “standard” horizon, flatness and monopole puzzles [1].

In the class of fundamental theories describing Planck scale physics (superstrings, quantum gravity) the natural scale, which sets the (nonvanishing) expectation values and masses of various fields, is the Planck scale itself,  $M_p \approx 10^{19}$  GeV. Hence, one has to face the problem of Planck scale potentials possibly playing, as we demonstrate below, a prominent role in the physics of the very early Universe. As important examples let us mention the so-called moduli fields arising in stringy models (they are allowed to have a potential when supersymmetry is broken), for instance the stringy dilaton which is believed to play an important role both in cosmology and in low-energy phenomenology and whose potential, perturbative or nonperturbative, remains undetermined.

Classical analysis shows that sufficient inflation is difficult to achieve in the classical potentials with the mass scales lying at  $M_p$ . Since one expects that the influence of quantum effects on the dynamics of the inflaton (by inflaton we understand any Planck scale field dominating the energy density of the Universe) and the metric is especially significant as the energy density approaches  $M_p^4$ , it is important to include quantum-mechanical effects into the analysis. Attempts have been made to describe the fluctuating inflaton field during the slow roll phase by solving explicitly the Fokker-Planck equation [2],[3]. In the stochastic approach, which we will adopt here, quantum fluctuations of the inflaton are investigated in phase space. This method has been applied to the usual chaotic potentials in [4].

In this note we follow the stochastic method and reexamine the evolution of the inflaton in phase space for the cosine potential, which has been analysed classically under the name “natural inflation” [5],[6]. We present the results of the numerical analysis based on stochastic dynamics [4] with the mass parameters of the “natural” potential both lying at the Planck scale. We choose homogeneous universes with varying cosmic kinetic energies. A minimal coupling between gravity and the inflaton field is assumed.

By means of stochastic dynamics, quantum fluctuations are given a statistical interpretation (quantum noise). The method takes into account the fluctuations in the local expansion rate due to the coupled evolution of the inflaton and the metric. The inflaton and its first time derivative (the inflaton “velocity”) are decomposed into two parts with respect to the physical horizon  $H^{-1}$  ( $H$  is the Hubble parameter): a long-wavelength (coarse-grained) part and a short-wavelength one. The quasi-classical field, whose evolution one follows, is formed by the coarse-grained

component, and the fluctuations arising in all Fourier modes are represented by the random force (the noise) in its equations of motion.

The coupled evolution equations for the coarse-grained components of the inflaton  $\phi$  and the velocity  $v = d\phi/dt$  are given by

$$\begin{aligned}\dot{\phi}(t) &= v(t) + \frac{H^{3/2}}{\sqrt{8\pi^2}} \eta(t) \\ \dot{v}(t) &= -3Hv(t) - V'(\phi) \\ H^2 &= \frac{8\pi}{3M_p^2} \left[ \frac{1}{2} v^2(t) + V(\phi) \right],\end{aligned}\tag{1}$$

where the prime denotes  $d/d\phi$ . The first equation has the form of a classical stochastic equation of motion (Langevin-equation) [7], where the random noise  $\eta(t)$  includes the effects of the rapidly fluctuating Fourier modes. This random noise term is missing in the classical analysis.  $\eta(t)$  is Gaussian random with zero mean and a correlation function  $\langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$ , which shows the Markov property of the process. In the approximation we work all spatial correlations in the random force are neglected to make the problem tractable (also allowing us to interpret ensemble averages as space averages). The cosine potential is given in terms of the parameters  $\Lambda$  and  $f$

$$V(\phi) = \Lambda^4 \left[ 1 + \cos\left(\frac{\phi}{f}\right) \right].\tag{2}$$

For simplicity we use  $\Lambda = f = M_p$ , although the analysis stays valid for any set of  $M_p$ -magnitude scales. The coarse-grained parts of the fields,  $\phi$  and  $v$ , are interpreted as classical random variables, whose evolution is modified by the influence of the stochastic noise. The average over a large number of independent realizations of the trajectories of the coarse-grained component is considered to be the classical homogeneous background. Treating the problem in phase space with the two independent phase space variables  $\phi$  and  $v$  has the advantage that initial conditions with a dominating kinetic energy contribution,  $\frac{1}{2}\dot{\phi}^2 \gtrsim V(\phi)$  are allowed. The method makes it also possible to control explicitly the fluctuations at the initial and final stages of inflation. There exists a well known procedure of solving a set of Langevin-equations by computer simulations, the molecular dynamics method [7]. Although the accuracy of the method is not very good (it is of the order  $\tau$  after  $N$  steps of integration;  $\tau$  is the step length), it provides a good estimate for the behaviour of the mean trajectory in phase space which is finally compared with the classical evolution. Usually, Itô's definition of the stochastic noise is used [7]. The

discrete evolution equations then take the form

$$\begin{aligned}
\phi_{n+1} &= \phi_n + v_n \tau + \frac{H_n^{3/2}}{\sqrt{8\pi^2}} w_n \sqrt{2\tau} \\
v_{n+1} &= v_n - 3H_n v_n \tau - V_n' \tau \\
H_n^2 &= \frac{8\pi}{3M_p^2} \left[ \frac{1}{2} v_n^2 + V_n \right],
\end{aligned} \tag{3}$$

where the index  $n$  characterizes the  $n$ th step of integration. The stochastic noise is simulated by a random number generator producing uniform deviates. They are turned into Gaussian deviates  $w_n$  with zero mean  $\langle w_n \rangle = 0$  and unit variance  $\langle w_n w_{n'} \rangle = \delta_{nn'}$  by the Box–Muller method [8]. Simultaneously, the iterations are performed using a different random number at each step. Finally, the average for a large number of realizations is taken.

The choice of a different interpretation of the stochastic noise is usually expected to have only a small influence on the final result [4]. Nevertheless, we perform the integration according to Stratonovich’s rule to show that deviations are possible. In the Stratonovich case the deterministic drift term  $v$  in the first equation of the system (1) acquires an additional contribution, the noise-induced drift:

$$D_{noise}^{(1)} = \frac{1}{2} \frac{\partial}{\partial \phi} D_{\phi\phi}^{(2)} = \frac{1}{2} \frac{\partial}{\partial \phi} \frac{H^3}{8\pi^2} = \frac{3}{16\pi^2} H^2 \frac{\partial H}{\partial \phi}. \tag{4}$$

$D_{\phi\phi}^{(2)}$  is the only nonvanishing entry of the  $2 \times 2$ -diffusion matrix. Using the Stratonovich rule, the first equation of (3) has to be replaced by

$$\phi_{n+1} = \phi_n + v_n \tau + D_{noise,n}^{(1)} \tau + \frac{H_n^{3/2}}{\sqrt{8\pi^2}} w_n \sqrt{2\tau}. \tag{5}$$

We present the results of the numerical integration of the system (3) in the dimensionless variables

$$x \equiv \frac{\phi}{\Lambda}, \quad y \equiv \frac{v}{\sqrt{2}\Lambda^2}, \quad \tau \equiv t\Lambda \tag{6}$$

for the cosine potential given by (2). In Fig. 1 the evolution of the stochastic mean trajectories (according to Itô’s (I) and Stratonovich’s (S) interpretation of the stochastic noise) are compared with the classical evolution of the inflaton in phase space. In the example we present here, the initial conditions are chosen in such a way that the total energy density of the inflaton is  $2\Lambda^4$  and the field is positioned at  $\phi/f = 3\pi/2$ . This initial value of the field is far away from the region around the maximum of the potential, where, by some fine tuning, one can enhance the number of e-folds in the classical case. No fine tuning of the initial conditions is required in our case. The mean trajectories shown are averages over  $3 \times 10^4$  independent stochastic trajectories. They share three characteristic features: (i) significant deviations from the classical path occur close to the point where, they approach the

classical slow roll trajectory, (ii) the mean velocity tends toward zero before the oscillating regime is reached, (iii) the averaged phase space evolution clearly depends upon the interpretation of the stochastic noise. Stratonovich's definition (S) prefers higher energies (this feature agrees with some earlier observations [3]). The general behaviour of the two mean trajectories is however similar.

The ratio of the noise-induced drift to the classical drift,

$$\mathcal{R} \equiv \frac{D_{noise}^{(1)}}{D_{class.}^{(1)}} = \frac{3}{16\pi^2 v} H^2 \frac{\partial H}{\partial \phi} = \frac{1}{\sqrt{6\pi} M_p^3 v} \sqrt{\frac{1}{2} v^2 + V(\phi)} \frac{dV(\phi)}{d\phi} \quad (7)$$

is also shown in Fig. 1. Here the equation (4) has been used together with the equality  $D_{class.}^{(1)} = v$ . The noise-induced drift acts always opposite to the classical one during the stage of the classical slow roll, so that the *averaged* evolution is being decelerated until the mean velocity vanishes.

The averaged quantities are not the whole story. One has to consider the distribution of the individual realizations around the mean. This is shown in Fig. 2 and Fig. 3 for the field and the corresponding velocity according to the Itô's rule. The numbers of integration steps are indicated. The step length was chosen to be  $\tau = 10^{-2}$ , so that 100 steps are equivalent to one Planck time  $M_p^{-1}$ . The steps are also given in Fig. 1 for the classical trajectory.

At the beginning,  $t \simeq \mathcal{O}(M_p^{-1})$ , when diffusion dominates over other dynamical effects, the distribution of the  $\phi$ -values around the mean remains Gaussian in its shape. Gradually, the  $\phi$ -dependent (nonlinear) drift coefficients cause deviations from the Gaussian distribution, which get significant at the time when the classical evolution would have already entered the epoch of coherent oscillations. It is clearly visible that at this stage the field has acquired values ranging over several minima of the potential. The maxima of the distribution are located near the minima of the potential. Diffusive processes are able to make the field drop into several different vacua.

The  $v$ -distribution is also purely Gaussian at the beginning although this is not shown in Fig. 3. The nonlinearity of the potential gives rise to significant deviations from the Gaussian shape already long before the initial velocity has decayed. It is remarkable that the distribution of this variable differs from Gaussian statistics considerably earlier than that of the  $\phi$ -variable. Gradually, it develops a tail toward positive velocities. At this stage the mean velocity constantly decreases, whereas the tail forms a second maximum, so that the distribution becomes symmetric with respect to the mean. This is not unexpected since the quantum "kicks" appear with similar probability in opposite directions (in the  $\phi$ -space), so that upward and downward motions are generated in equal abundances. One should note that the total width of the  $v$ -distribution is much smaller than that of the  $\phi$ -distribution.

The final stage of the stochastic evolution is characterized by a zero mean velocity. This is due to the symmetry of the  $v$ -distribution. The averaged drift is zero, so that the averaged field seems to stick to one value. However, the most probable

values of the velocity are by no means zero (thus indicating the dominance of the fluctuations in the inflaton dynamics); positive velocities are then equally probable as negative velocities, so that the noise-induced movement uphill has nearly the same probability as the downward motion. We have found differences between the time evolution of the inflaton for Itô's and Stratonovich's interpretations of the stochastic noise. But both cases share the same basic feature: since the noise-induced drift always acts against the classical drift, the time evolution of the mean quasi-classical field (compared with the classical case) can be delayed by quantum fluctuations. The stochastic analysis shows that the duration of the quasi-de Sitter phase can be prolonged to reach a large number of e-folds, much larger than 60. The *a priori* probability for this to happen is estimated to be of the order of one (see discussion below).

The behaviour described so far is generic for any Planck scale field dominating the energy density of the Universe. The discussion demonstrates clearly that the classical description of the cosmological evolution driven by such a field is highly inadequate and that a prolonged quasi-de Sitter phase is naturally generated in the quantum description.

Let us try to understand whether this quantum inflation can provide us with a model for successful inflation, solving cosmological puzzles. The basic requirement for the scenario under consideration to serve as a successful inflationary model is that the number of e-folds produced during the superluminal stages of the evolution of the Universe must be large enough, larger than 60 at least. Of course, there are model universes in the ensemble where the inflaton rather rapidly approaches values near the minimum of the potential and then stays there. In this case the generated number of e-folds is insufficient.

However, there is a large enough fraction (of order unity) of all observed realizations of the scenario where the fluctuations act in such a way that the field varies (over macroscopic periods of time) much more slowly than required by the classical evolution. In this case, a sufficiently large number of e-folds could be generated to solve the cosmological puzzles. Fig. 4 shows the development of the number of e-folds produced during the evolution of the inflaton for one typical stochastic path compared with the classical result. Apparently, the quantum dominated motion easily provides more than 60 e-folds. This means that an inflationary epoch is generated by quantum fluctuations for a generic inflaton potential with parameters fixed at the Planck scale. Our results indicate that even the *a priori* probability of quantum inflation is large, not to mention the *a posteriori* probability [6], which is exponentially close to one. We stress again, that the prolonged quasi-de Sitter stage is a generic feature of our scenario and does not require a fine tuning of the initial conditions.

The end of the inflationary period and the reheating of the Universe in this scenario have to be examined more carefully. Generally, the strength of the fluctuations increases with increasing values of the potential (as the Hubble parameter increases), and the field can easily be driven uphill thus leading to a prolongation

of the quasi-de Sitter phase. Hence, the possibility of "eternal" inflation [9] is incorporated in this model. A natural exit is always possible with the assumption of the instability of the inflaton. By decaying into light particles a sufficient reheating temperature necessary for baryogenesis could be provided and inflation be stopped. A specific feature of this violently fluctuating system is the non-zero probability of large fluctuations throwing the field out of the inflationary regime thus ending abruptly the quasi-de Sitter phase.

One possible problem of quantum Planck scale inflation is the excessive production of energy density inhomogeneities. Since the relative magnitude of a physical length scale to the horizon,  $\lambda/H^{-1} \propto \dot{R}$ , is a fluctuating quantity, any scale  $\lambda$  crosses the horizon not only once, but frequently within the inflationary period. For this reason, the resulting amplitude of generated density perturbations need to be calculated carefully, as it would serve as a very sensitive probe of the present scenario. One obvious way out of the density fluctuations problem is to assume that the final stage of inflation, when density perturbations at observable scales are produced, is controlled by some other, lower-scale, field (like in multiple-inflaton models [9]), which can easily be the case in models, like string inspired ones, where the number of relevant degrees of freedom is rather large. In such a case, quantum inflation is the natural trigger for the subsequent inflationary epoch and can easily provide a quasi-de Sitter stage of realistically long duration.

On the basis of our experience with a wide class of potentials (we have found similar effects for the quadratic and quartic potentials of chaotic inflation) we want to conclude that the presence of quantum inflation seems to be a generic feature of field theoretical models addressing Planck scale physics and it may be interesting and fruitful to explore this phenomenon in more detail.

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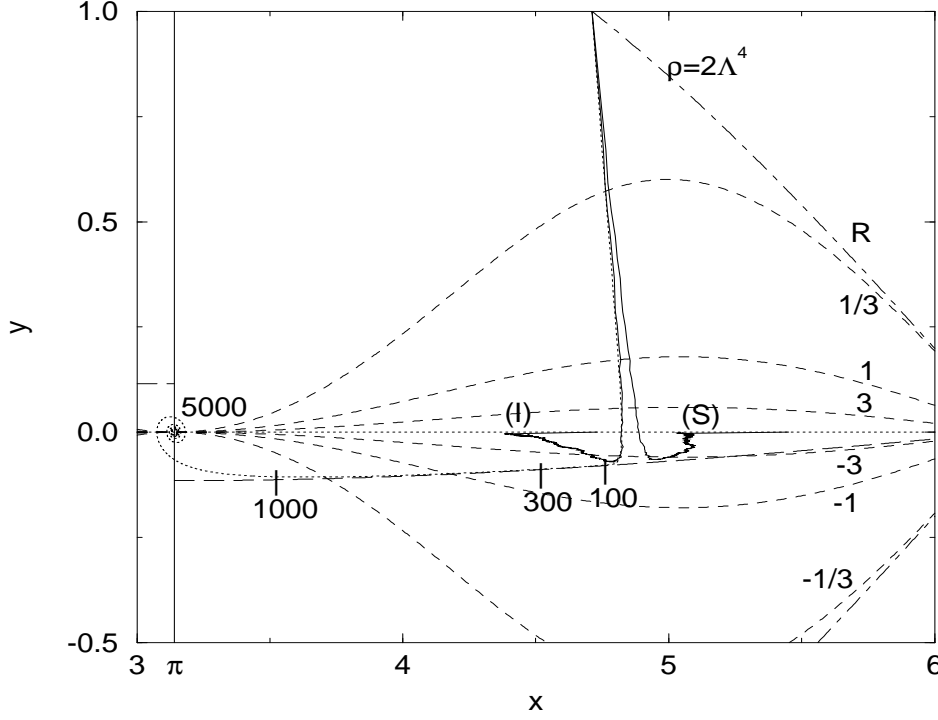


Figure 1: Stochastic mean trajectories (averaged over  $3 \times 10^4$  realizations) in the potential  $V(\phi) = \Lambda^4 [1 + \cos(\phi/f)]$  according to Itô's (I) and Stratonovich's (S) interpretation of the stochastic noise. The parameters  $\Lambda$  and  $f$  are set to the Planck scale. The classical trajectory is shown by the dotted curve. It meets the slow roll curve (long dashed) after about 30 steps of integration. The initial condition is chosen to lie on the curve of constant energy density  $\rho = 2\Lambda^4$  (dot-dashed lines) with a field value of  $\phi/f = 3\pi/2$  and positive velocity.  $3 \times 10^4$  integration steps were performed with a step size of  $\tau = 10^{-2}$ . Several lines of constant ratio  $\mathcal{R}$  (dashed lines) are shown. The absolute value of  $\mathcal{R}$  increases, if one moves towards  $y = 0$ . The values of  $\mathcal{R}$  and the number of integration steps along the classical trajectory are indicated.



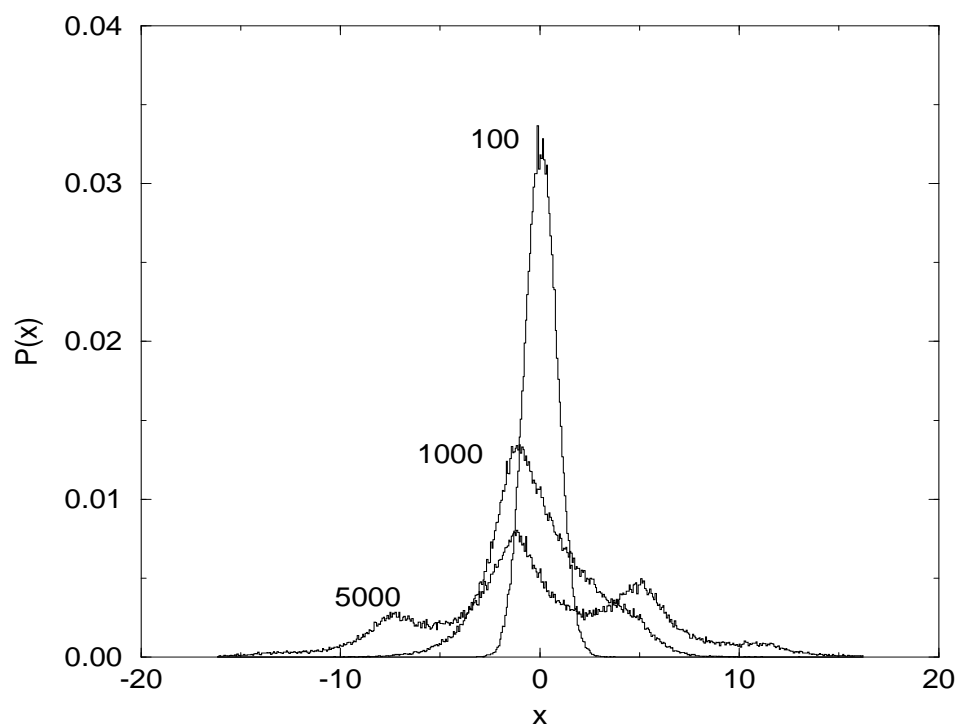


Figure 2: Distribution of the  $\phi$ -values of  $10^5$  realizations (normalized) around the mean ( $x=0$ ) after 100, 1000 and 5000 steps of integration. The influence of the potential is visible.

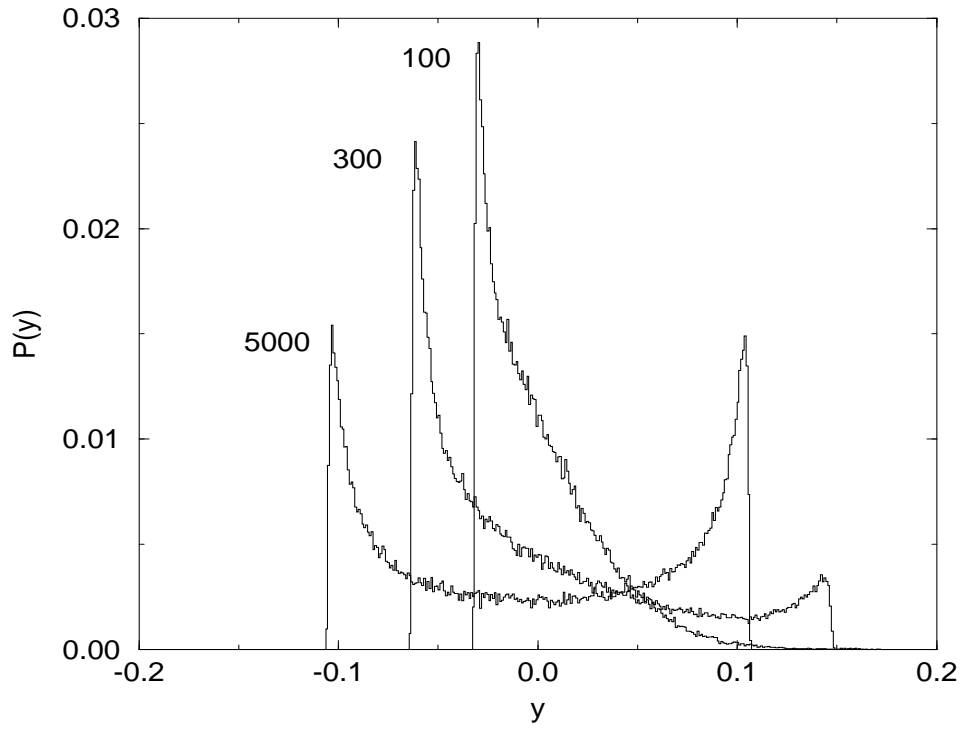


Figure 3: Distribution of the  $v$ -values of  $10^5$  realizations (normalized) around the mean ( $x=0$ ) after 100, 300 and 5000 steps of integration.

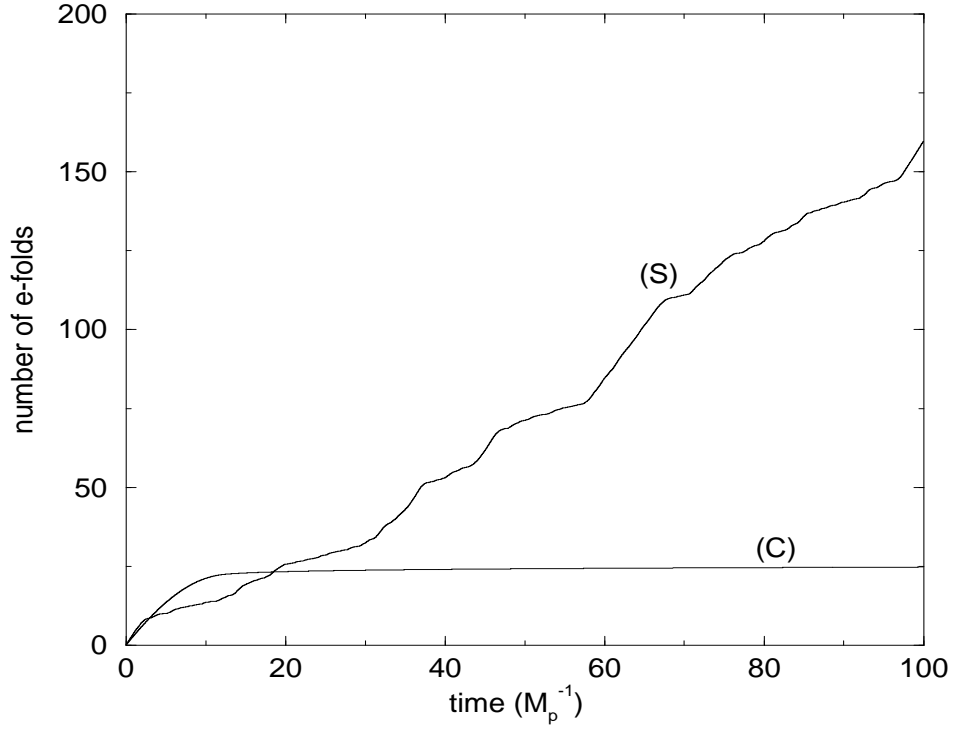


Figure 4: Total number of e-folds produced during the evolution of the inflaton. The initial condition and the parameters of the potential are chosen as in Fig. 1. The classical analysis (C) clearly shows the end of inflation after the field has reached the minimum of the potential. Only 25 e-folds can be produced before the entry into the hot Friedmann universe. The time behaviour of a typical stochastic solution (S) reveals a continuous superluminal expansion of the universe; the demanded number of 60 e-folds is easily exceeded.

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